Networking between kinds of software: geometric loci and exploration of simple models of space curves *

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Abstract

Geometric loci are an important topic in geometry. We explore constructs of plane curves and of space curves inspired by space trajectories (bicircular and tricircular motions). The exploration relies on joint usage of Dynamic Geometry Software (DGS) and of a Computer Algebra System (CAS), and involves automated methods to determine loci and sometimes dichotomy methods. The computer assisted work reveals some surprises, in particular the respective roles of the two kinds of software may be different from what previous works revealed. If previously the roles were quite distinct, DGS for exploration and numerical approach, and CAS for symbolic proof, here the CAS is needed for numerical methods also. The curves are described by trigonometric parametrization, and implicitization is performed using elimination, for 2D and 3D models. Finally, we discuss approaches to developments using models and elaborate on the importance to develop automatic dialog between kinds of software.

1 Introduction

1.1 A motivation: frequent presentation of transfer orbits of space probes.

A modern way to attract students to study mathematics consists in proposing activities around items from the daily news: for example, events related to the International Space Station (ISS), the almost simultaneous launching of three Mars-bound spacecrafts in February 2021 and their arrival 6 months later, the James Webb space telescope launched 10 months later, etc. Since then, all of these provide big titles. More recently, the race to the Moon, American Artemis lunar program, the Chinese mission to the far side, attract much attention. The general audience discovered that a spacecraft's trajectory from the Earth to Mars is not straight but curved (Figure 1).

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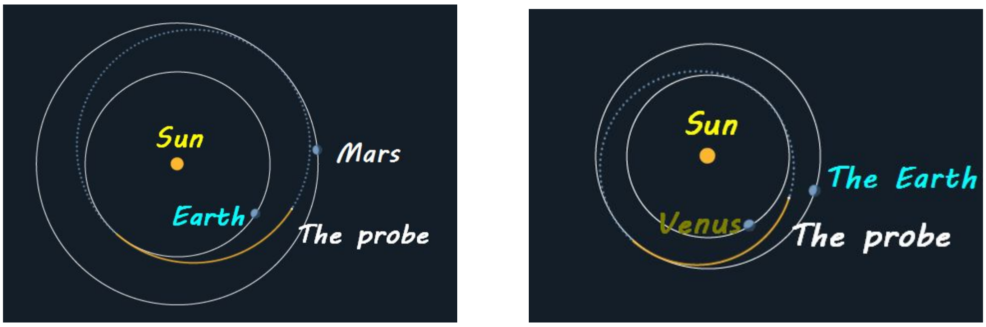

Figure 1: Transfer orbits between the Earth and another planet

Roughly speaking, each of these trajectories is the union of arcs of ellipses. The shape of the trajectory and the velocity of the spacecraft are ruled by the so-called Kepler laws (Karttunen et al., 2008). Kepler's 1st law reads that the trajectory of an object P orbiting an object S under the influence of gravitation only is an ellipse with S at one of the foci. Kepler's 2nd law explains how the velocity changes along the trajectory (see Figure 3): flying from C to D or from E to F takes the same time if, and only if, the curved triangles SCD and SEF have equal areas.

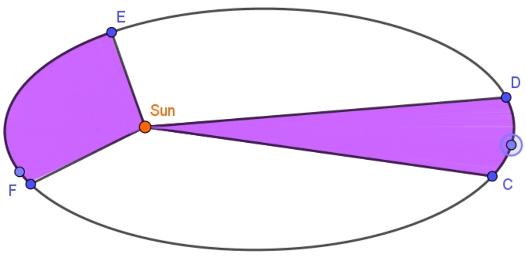

Figure 2: Kepler laws.

The Israeli Moon-bound probe Beresheet was planned to move along ellipses. At specific times, thrust has been exploited to increase eccentricity, keeping the Earth as a focus, until the ellipse englobes the Moon. Then the probe has been slowed down to be captured by the Moon's gravitational field. At that time the trajectory is an ellipse with the Moon at a focus. The probe orbits te Moon in the original direction. Figure 3 shows a diagram of the planned trajectory.

Artemis's trajectory seems more complicated (Figure 4). The diagram on the left shows the points were thrust is operated to change the geometric characteristics of the trajectory. After the engines stop, the motion is again ruled by Kepler laws, and the trajectory follows an arc of a new ellipse with the Sun at a focus. At some time, the probe is slowed down to be captured by the gravitation field of the Moon. From now on, the orbit is an ellipse with the Moon as a focus. The figure on the left seems static, but the figure on the right expresses the fact that actually the Moon moves on its orbit around the Earth, pulling the spacecraft with it. Simple models in 2D and 3D (taking into account the fact that the Sun also moves) have been displayed in [13]

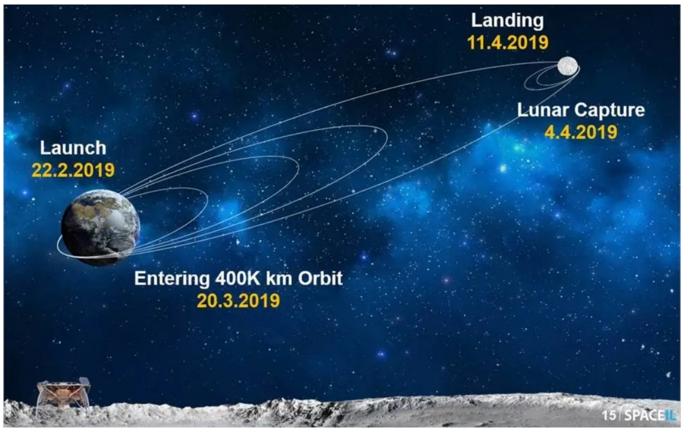

Figure 3: Beresheet's trajectory (Credit: NASA - SpaceIL)

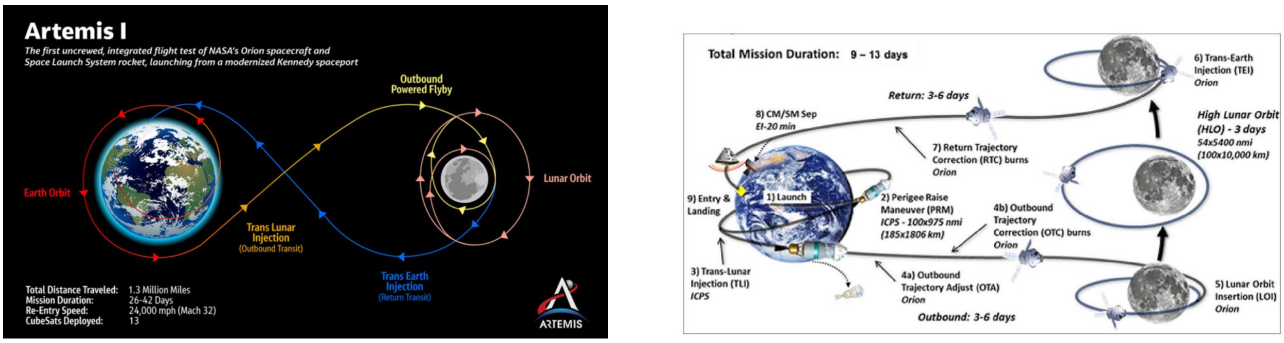

Figure 4: Diagrams of Artemis trajectory (Credit: NASA)

These two examples explain why we explore bicircular and tricircular motions, (a) when all the components are oriented in the same direction(Section 2) and (b) when one component is retrograde (Section 3).

All this is a little bit caricatural. In reality, more than one space object influence the trajectory, and the axis of a planet and/or a natural satellite may make some angle with the orbital plane. For example, the axis of the Earth makes an angle of $23^o27'$ with the ecliptic plane. So, the ecliptic plane and the equatorial plane of the moon may be different. This influences the geometric description of the trajectories. Moreover, for travelling from the Earth to the Moon, engineers have to take into account the gravitational influence of Mars also. As we are interested in simple models accessible to either upper-high school or to beginning undergraduates, we simplify all this.

According to Kepler laws, the planetary motion is well described by an equation in polar coordinates, namely

$$r = \frac{p}{1+\varepsilon\theta},$$

where r is the distance from the planet to the Sun, ε is the eccentricity of the ellipse and θ is the angle to the planet's current position from its position closest to the Sun (called aphelion). Early undergraduates' literacy with polar coordinates is generally not so high. Therefore, we chose to work with parametric presentations for plane curves, which yielded the construction of, sometimes, classical curves such as epitrochoids, epicycloids, etc. The orbits of the planets of the Solar System are ellipses whose eccentricity is very close to 0, therefore for a 1st approximation, suitable for beginners, the orbits have been modeled by circles centered at the Sun, and velocity is constant, encoded in the angular velocity. Other simplifications were as follows:

- The mean distance Sun-Earth is equal to 1 (astronomical unit, A.U.), whence the definition of E at the beginning of Section 2.
- The period of the Earth on its orbit is 1 (same remark as above).
- The orbital data for another planet is given in proportion to these data. We caplin in next section why we decided not to use actual orbital data.

Actually, the only "true" orbital data we can be interested in consists in distance between Sun and planets, or between planets and their satellites, either natural or artificial. At the beginning of next section, we explain why this is irrelevant to the present work. The author teaches in an advanced course about satellites and Earth observation. There, the "true" shape of orbits, (whence variable velocity, etc.), inclination with respect to the equator, are considered. Anyway, the intuition acquired with the simple models presented here helps.

1.2 What are we doing, what are the educational goals?

The ratio between the distances Sun-Earth and Earth-Moon is close to 413. No hope to present on a screen the Sun and trajectories, neither of the Moon around the Earth, nor of an object orbiting the Moon. Therefore, we build the activities with arbitrary values of the parameters, not with genuine orbital data. Nevertheless, these choices provide constructions of interesting curves, from which the student can imagine what happens in space. Most of the examples in [13] are planar, even when taking the Sun's motion into account: the Sun, the planet and the satellite move in the same plane (in

Section 2). Animations and computations are performed with a Dynamic Geometry Software (DGS, here GeoGebra) and a Computer Algebra System (CAS, here Maple 2024), and networking between them. For one example, we compute an implicit equation for the trajectory, using Elimination. We provide Maple session's code.

In Section 4, we consider a model of a satellite moving around the planet on an orbit not included in the plane of the planet's orbit, here a polar orbit. In this last case, we begin working with parametric presentations for the orbits and implicitize the presentation. As for any space curve, an implicit presentation shows it as the intersection of two surfaces (i.e. we look for 2 polynomial equations). This provides a new opportunity to compare the affordances of plotting with a Dynamical Geometry Software (DGS) and with a Computer Algebra System (CAS), comparing parametric plots and implicit plots.

The activities that we propose are based on exploration and discovery in a technology-rich environment. The outcome is multi-faced: plots based on numerical data, search for exact (symbolic) expressions, implicitization, etc. Some of the activities with DGS provide some knowledge, but further exploration has to be performed with the CAS, among other issues, analyzing numerical solutions of a trigonometric equation (Section 3). The core of the search is based on thinking and computing in a way, which has been learnt by undergraduates in a 1st Calculus course.

2 Tricircular motion: a satellite on a direct orbit around the Moon in the SUN-Earth-Moon plane

We consider a moving point E given by $(x_E, y_E) = (\cos t, \sin t)$. An object M orbits E according to

$$(x_M, y_M) = (\cos t + r\cos(ht), \sin t + r\sin(ht))\,, \tag{1}$$

where r is the distance from E and h encodes the orbital period[1].

Now, we define an object Sat by the following formula:

$$(x_{Sat}, y_{Sat}) = (x_M, y_M) + r/2(cos(3hu), sin(3hu)). \tag{2}$$

The orbit of Sat looks more complicated in any case. A GeoGebra applet [S1] is available at https://www.geogebra.org/m/mnsdv3fz; the reader is invited to explore with the sliders. The orbits can be visualized (a) with the command **Locus** and (b) running an animation with **Trace On**. The locus of E is a circle and the orbit of M is an epicycloid[3]. Examples are the dotted curves in Figures 5,6 and 7.

Figure 5 shows 3 cases, for $r = 0.2$ and $h = 3, 5, 6, 8, 9$ (from left to right).

For for $r = 0.4$ and $h = 3, 4, 5, 6, 8$ (from left to right), we have Figure 6, and for $r = 0.8$ and $h = 3, 4, 5, 6, 9$ (from left to right), we have Figure 7:

At first glance, we conjecture that for odd h, both the $x-$axis and the $y-$axis are axes of symmetry, but for even h, only the $x-$axis is such. Of course, this requests a symbolic proof. Rotational symmetries are not very apparent, but may exist. Broader exploration can be made with the above

[1]For the sake of simplification, we consider here only integer values for h. For non integer values, determining the motion periodicity is harder. Such examples will be studied in a f[2]urther work.

[3]See https://mathcurve.com/courbes2d.gb/epicycloid/epicycloid.shtml.

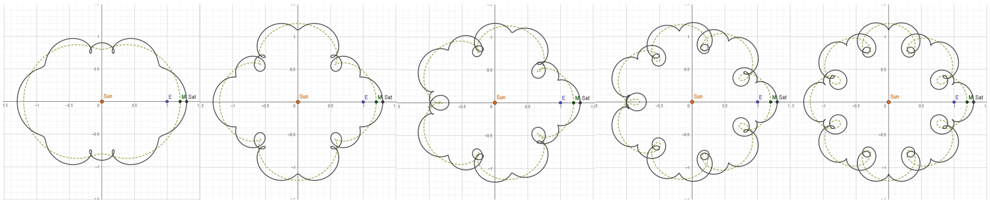

Figure 5: Three coplanar movements in the same direction - $r = 0.2$

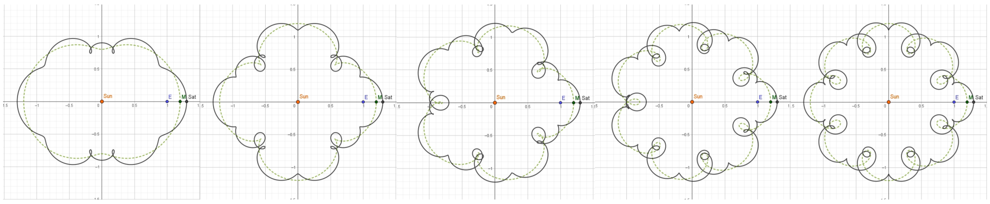

Figure 6: Three coplanar movements in the same direction - $r = 0.4$

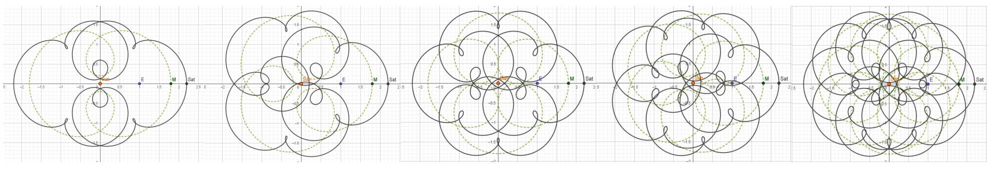

Figure 7: Three coplanar movements in the same direction - $r = 0.8$

mentioned applet.Note that both trajectories, M viewed from the Sun and Sat viewed from E, are epicycloids. The composition of the motions yield the complicated curves.

The comparison of two different points of views is interesting. On the basis of Tycho Brahe's observations, Kepler drew a sketch of the orbit of Mars viewed from the Earth [22]; the curve that he plotted (Figure 8) is very similar to curves obtained in other situations (see [7]). In an other direc-

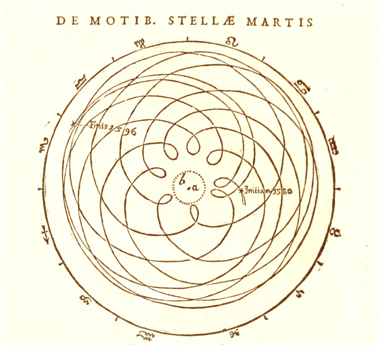

Figure 8: Kepler's sketch of Mars's orbit when viewed from the Earth

tion, the website `https://mathcurve.com/courbes2d.gb/courbes2d.shtml` shows constructions of epicycloids by means of a circle rolling on another circle. We can construct the complicated curves obtained in this section using a circle rolling on a circle rolling on a 3rd circle. This is beyond the scope of the present paper. By the time these rows are written, the author is waiting for the outcome of students' activities.

3 A satellite on a retrograde lunar orbit in the SUN-Earth-Moon plane

3.1 Equations and first examples

Here too, 3 different coplanar motions compose the satellite motion, when viewed from outside: the planet E, its moon M and the satellite, all of them described by trigonometric parametric presentation. The difference with the previous section is that here the satellite moves in retrograde direction. Recall Artemis trajectory, where the probe orbits the Moon in retrograde direction (Figure 4). Models for the composition of two direct motions and a retrograde 3rd motion have already been described in [13]. It provides non classical curves, as shown in Figure 10. The satellite moves according to the equation

$$(x, y, z) = (\cos u, \sin u) + r(\cos au, \sin au) + s(\sin bu, c \cos bu); \quad u \in [0, 2\pi]. \tag{3}$$

WLOG, we chose to work with coefficients 1/3 and 1/2, other choices could provide curves with similar properties. For the mathematical work, the given interval can be replaced by the whole of $\mathbb{R}$, then the parametrization determines infinitely many copies of the same geometric locus. But working with software requests a closed interval. With a suitable choice of the interval, and with the *increasing* option for animation with a slider, GeoGebra mimics infinitely many periods on the orbit.

The retrograde motion of the 3rd object is encoded by the last term in the sum. A GeoGebra applet [S2] is available at https://www.geogebra.org/m/p7uaewbc.

For $a = 1$, the motion is actually bicircular and the curve is a hypotrochoid[4].Examples with $r = 1/3$ and $s = 1/2$ are displayed in Figure 9, for $b = 3, 4, 5, 6, 9$.

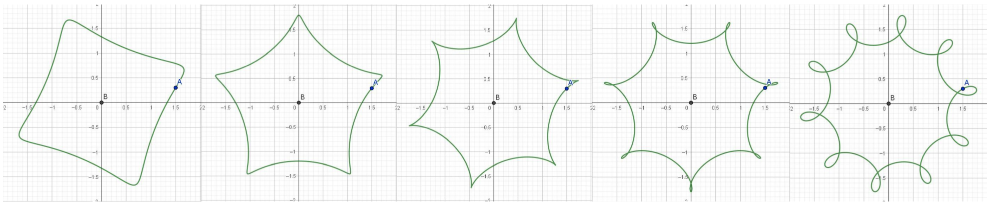

Figure 9: Three movements, one in retrograde direction - classical curves

The reader is invited to play with the 5 parameters involved and explore the existence of self-intersections and of cusps. In [13], we show various examples, including the so-called Mystery Curves, and apply them for creating Math Art and 3D printed objects[5]. Another point of view on the Mystery Curves can be found in [4]. Figure 10 displays the output for $r = 1/3$, $s = 1/2$ and $(a, b) = (4, 8), (6, 14), (8, 14), (8, 17)$ from left to right. The two curves on the right show also a partial view on how the animation is displayed.

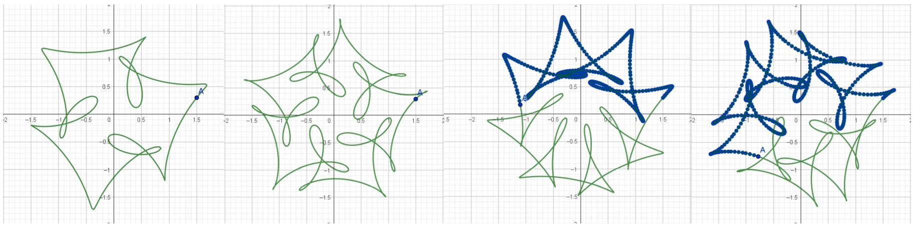

Figure 10: Three movements, one in retrograde direction

3.2 Exploration of the shape of the curves

Using the different sliders to make the values of the parameters vary, provides activities with different kinds of focus, according to teacher's and students' creativity. One of them is to study the symmetries of the curves. Rotational symmetries can appear, but the greater the parameters a and b, the harder to understand are the curves and rotational symmetries may remain unidentified. Generally the curves present one or two axial symmetries, but not more. Rotational symmetries of order 5 appear for other choices of the parameters, e.g., for $(a, b) = (6, 9)$. All this is purely experimental, but in this last case, computations with a CAS may be possible. Note that the explorations described here may provide

[4]See the Mathcurve page at https://mathcurve.com/courbes2d.gb/hypotrochoid/hypotrochoid.shtml.

[5]Animations can be found in [14].

conviction, but it is not a proof (for detailed analysis of the difference, and of what should be done, refer to the book [19]).

Let us study one example. Figure 12 corresponds to $(r, s) = (0.5, 0.3)$ $(a, b) = 11, 4)$. The curve seems to have a rotational symmetry of order 5. This can be explored in various ways. One of them can be to prove that $u \in \mathbb{R}$, $Sat(u + 2\pi/5) = Sat(u)$, but this may be unilluminating, actually only numerical, as there is no simple closed formula for these sines and cosines.

3.3 A 1st exploration - symmetry

On the basis of the conjecture that there is a 5-fold rotational symmetry, we consider one arc of the curve, given by

$$(x, y, z) = (\cos u, \sin u) + \frac{1}{3}(\cos au, \sin au) + \frac{1}{2}(\sin bu, c\cos bu); \quad u \in \left[0, \frac{2\pi}{5}\right]. \tag{4}$$

The corresponding arc can be plotted with GeoGebra using the parametric presentation and the command **Locus(Point creating the locus, slider)**. Then the arc should be rotated about the origin by an angle of $2\pi/5$. As the locus appearing on the screen is not recognized by the software as a geometric construct, the automatic command for rotation does not work. The point A defined by Equation (4) has to be rotated, using GeoGebra's button for rotations, and is automatically identified as depending on the parameter (the slider) u. Now the **Locus** command works. The process has to be iterated 4 times. The arcs are shown in Figure 11 with different colors and dots. A GeoGebra applet [S5] is

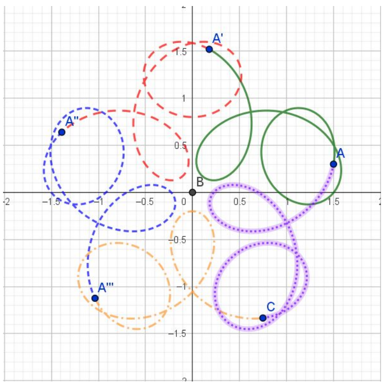

Figure 11: Exploration of the 5-fold symmetry of one of the curves

available at `https://www.geogebra.org/classic/vhdkpfgk`.

This process can be adapted to every case where a rotational symmetry seems to appear. Exploration with variable parameters (a, b) provides numerous examples with rotational symmetries of exotic order (we mean not the simple cases generally shown in class, but order 7,11, 13 etc.). Extension of such constructs may provide interesting mathematical art, as in [7, 13].

3.4 A 2nd exploration - circumcircle

Here, the exploration is visual only. The Locus command provides a plot based on numerical data, and the locus on the screen is not a geometrical construct. Therefore, the button and the intersection commands do not work here. Until now, we do not have a symbolic equation either. As the coordinates are sums of sines and cosines, they are bounded. Is it possible top find a circumcircle? The situation is illustrated in Figure 12.

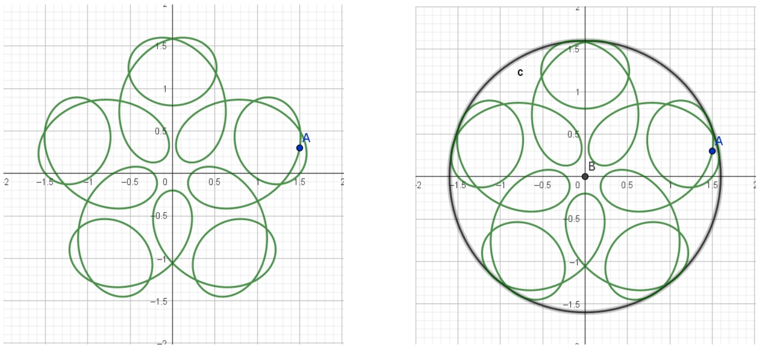

Figure 12: The curve has a rotational symmetry of order 5

The number of points of intersection of the curve with a circle centered at the origin can give an interesting indication. The exploration yields a radius approximatively equal to 1.6 fro a circumcircle. In order to check this, it is necessary to solve the following equation

$$x(t)^2 + y(t)^2 = 1.6^2 \tag{5}$$

for the unknown t. Hand-made computations are unilluminating, whence the importance to use a CAS. Even with technology this is not trivial. After more than one hour on a recent computer with a strong CPU, Maple did not provide an answer. Further attempts have been performed for $(a, b) = (6, 9)$. Experimentation with GeoGebra suggests that the curve is circumscribed by a circle centered at the origin with a radius R between 1.7 and 1.8. Further experimentation has been made with Maple, solving equations similar to Eq. (5) for variable radius. For each tested value of the radius, Maple gave an immediate answer.

Experimentation provide the following output:

- For $1.7 \leq R \leq 1.73$, Equation (5) has 10 real solutions and numerous complex non real solutions. This corresponds to 10 points of intersection of the circle with the curve $\mathcal{C}$. The GeoGebra plot reveals that the circle is not a circumcircle to $\mathcal{C}$.

- For $R \geq 1.74$, Equation (5) has only non real solutions. The circle does not intersect the curve $\mathcal{C}$. A approximation of the radius can then be found numerically by a dichotomy method (similar to the dichotomy method learnt by undergraduates in a 1st Calculus course[6])

A similar process can be applied with GeoGebra. The radius of the circle has to be defined by a slider, and the increment of the slider can be defined as smaller than the default one. The precision of the work depends here on te user's hand.

[6]This is a good opportunity to enhance the fact that learning ways of thinking is as important as learning new notions and theorems.

4 A satellite on a polar lunar orbit

4.1 With "fixed Sun"

Because of the Earth revolution about its axis, a satellite orbiting the Earth on a polar orbit may provide imaging of the entire Earth surface, whence its importance for Earth observation. Details are to be found in [5]; an online simulator is available at https://observablehq.com/@jake-low/satellite-ground-track-visualizer. Here, for visualization and animations, the model requests usage of a software for 3D geometry. We use GeoGebra's 3D Graphics package. The Sun is placed, as usual, at the origin. A "planet" E is defined by the following equations (the coefficient 2 has be introduced in order to have "reasonable" proportions on the plot):

$$\begin{cases} x_E(t) = 2\cos t \\ y_E(t) = 2\sin t \\ z_E(t) = 0 \end{cases} . \tag{6}$$

and a moon M in the plane of the orbit of E by

$$\begin{cases} x_M(t) = 2\cos t + 0.4\cos at \\ y_M(t) = 2\sin t + 0.4\sin at \\ z_M(t) = 0 \end{cases} , \tag{7}$$

where a is a positive parameter. The GeoGebra commands are as follows:

```
E=2(cos(u),sin(u),0)
M=E+0.4 (cos (au),sin(au),0)
Sat=M+0.4(cos(bu),sin(bu),0)
```

Sliders for a and for b have to be defined; here, we did it for integer values (otherwise, the periodicity issue becomes hard to deal with - we intend to addres this issue in a further work). The coefficients 0.4 and 2 have been chosen in order to have a readable plot. We mentioned already that the ratio between true distances between Sun, Earth, Moon, and artificial satellites in the Solar System do not enable to display faithful models on a computer screen.

Now we define a satellite Sat around M orbiting in a plane perpendicular to the planes of the orbits of E and M.

$$\begin{cases} x_{Sat}(t) = 2\cos t + 0.4\cos at \\ y_{Sat}(t) = 2\sin t + 0.4\sin at \\ z_{Sat}(t) = 0.4\cos bt \end{cases} . \tag{8}$$

Figure 13 shows the curves obtained for different values of the parameters. These are snapshots of a GeoGebra applet [S3], available at https://www.geogebra.org/m/guwguzgk. On the left are displayed the orbits of E and M, on the right the 3D situation is shown, including the orbit of Sat. It is clear that the projection of the orbit of Sat on the $x, y-$plane is the orbit of M. This one is a epicycloid, a well-know curve (see https://mathcurve.com/courbes2d/epicycloid/epicycloid.shtml). For example, $(a, b) = (3, 2)$ gives a nephroid.

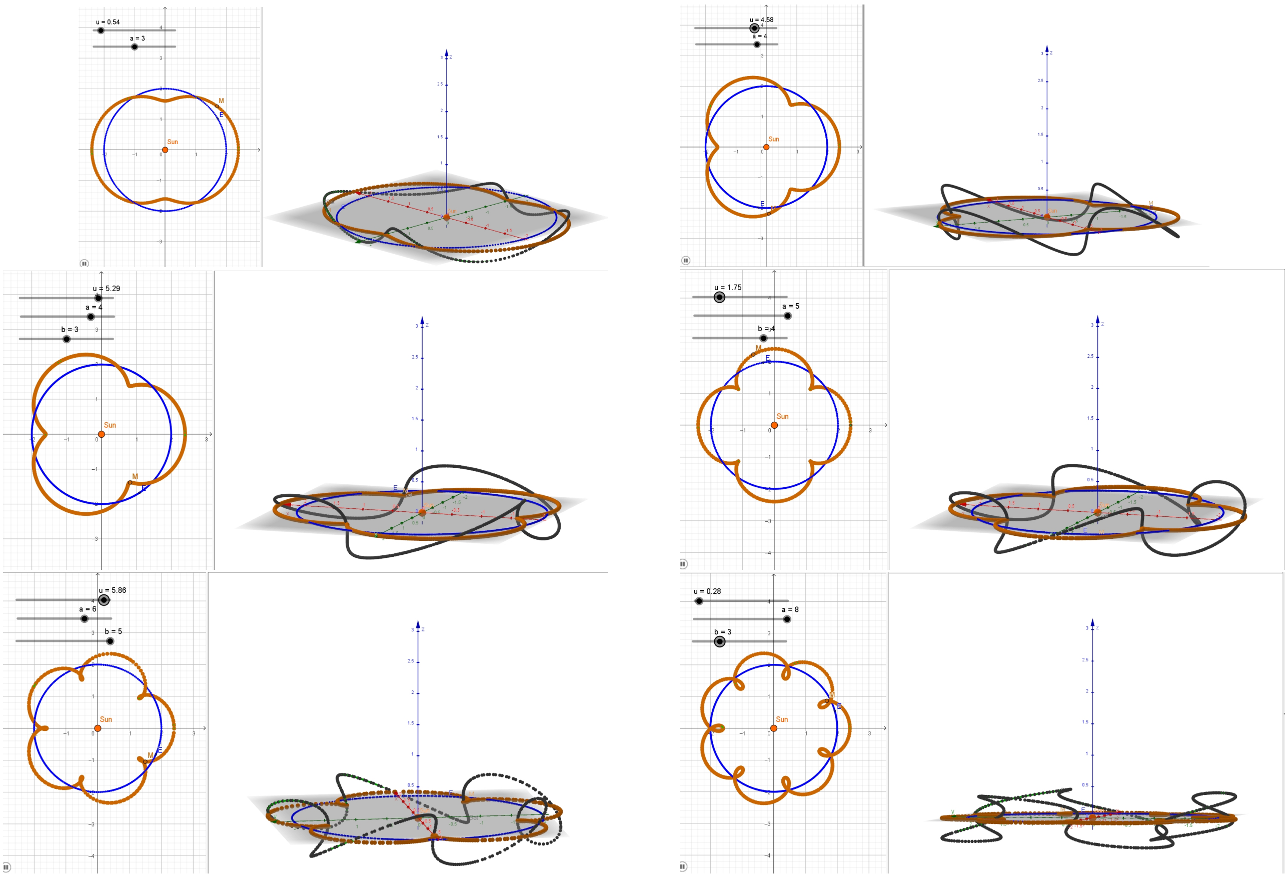

Figure 13: Polar orbits around a moon

4.2 With ”moving Sun”

For the sake of simplicity, we consider a ”Sun” moving according to the parametric presentation $(x, y, z) = (0, 0, u);\ u \in \mathbb{R}$. The only modifications to the commands are:

```
Sun=(0,0,u)
E=Sun+2(cos(u),sin(u),0)
```

The other points are changed automatically. Figure 14 shows two cases, corresponding to $(a, b) = (3, 2)$ on the left and to $(a, b) = (5, 6)$ on the right. The plane containing the orbits of E and M has been made apparent, and moves according to the Sun.

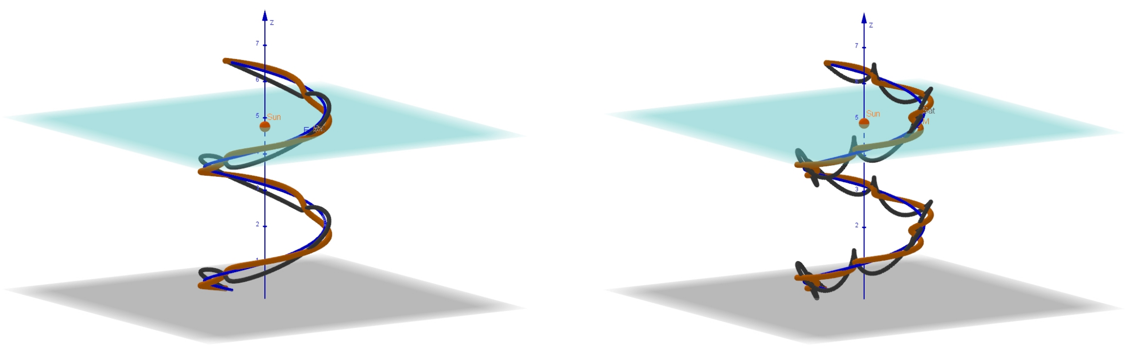

Figure 14: Polar orbits around a moon - a dynamic view with moving Sun

5 Implicitization

5.1 The process and the implicit equation – a planar situation

All the situations and activities presented above are based on trigonometric parametric equations for the different curves. For graphics, these provide accurate plots, as explained in [31]. For other issues, it may be better to have implicit equations. Therefore, books such as [30, 25, 29] and online encyclopedias of curves display both parametric and implicit equations (also polar equations), when possible. Implicitization is not an easy task, and requires heavy algebraic machinery. For our trigonometric parametric equations, sines and cosines have to be substituted by rational expressions, then polynomials can be defined. They generate ideals, for which algorithms from Gröbner bases theory and Elimination are generally used. For example, this has been done extensively in [6].

The situation given here is more complicated, as we deal with tricircular motion. As we saw in Equations (8), sines and cosine appear with 3 different angular velocities, which makes the substitution more intricate. We show here only one example.

The curve presented in Figure 12 has parametric equations

$$(x, y) = (\cos(u) + 0.5\cos(11u) + 0.3\sin(4u), \sin(u) + 0.5\sin(11u) + 0.3\cos(4u)) \tag{9}$$

It is well-known that for every $u \in \mathbb{R}$, it exists a $t \in \mathbb{R}$ such that

$$\begin{cases} \cos u = \frac{1-t^2}{1+t^2} \\ \sin u = \frac{2t}{1+t^2} \end{cases} . \tag{10}$$

On the one hand, using De Moivre formula, we have:

$$(\cos u + i \ \sin u)^4 = \cos 4u + i \ \sin 4u.$$

On the other hand, Newton binomial yields:

$$(\cos u + i \ \sin u)^4 = \cos^4 u + 4i \ \cos^3 u \sin u - 6\cos^2 u \sin^2 u - 4i \cos u \sin^3 u + \sin^4 u.$$

Comparing real parts and imaginary parts in bote equations, we obtain:

$$\begin{cases} \cos 4u = \cos^4 u - 6\cos^2 u \sin^2 u + \sin^4 u \\ \sin 4u = 4\cos^3 u \sin u - 4\cos u \sin^3 u \end{cases} . \tag{11}$$

Substituting here Equations (10) and simplifying, we obtain:

$$\begin{cases} \cos 4u = \frac{t^8-28t^6+70t^4-28t^2+1}{t^2+1)^4} \\ \sin 4u = \frac{-8t^7+56t^5-56t^3+8t}{(t^2+1)^4} \end{cases} \tag{12}$$

By the same method we obtain:

$$\begin{cases} \cos 11u = -11\cos u \sin^{10} u + 165\cos^3 u \sin^8 u - 462\cos^5 u \sin^6 u + 330\cos^7 u \sin^4 u \\ \qquad -55\cos^9 u \sin^2 u + \cos^{11} u \\ \sin 11u = -\sin^{11} u + 55\cos^2 u \sin^9 u - 330\cos^4 u \sin^7 u + 462\cos^6 u \sin^5 u \\ \qquad -165\cos^8 u \sin^3 u + 11\cos^{10} u \sin u \end{cases} \tag{13}$$

and finally[7]

$$\begin{cases} (t^2+1)^{11}\cos 11u = -t^{22} + 231t^{20} - 7315t^{18} + 74613t^{16} - 319770t^{14} \\ \qquad +646646t^{12} - 646646t^{10} + 319770t^8 - 74613t^6 + 7315t^4 - 231t^2 + 1 \\ (t^2+1)^{11}\sin 11u = 22t^{21} - 1540t^{19} + 26334t^{17} - 170544t^{15} + 497420t^{13} \\ \qquad -705432t^{11} + 497420t^9 - 170544t^7 + 26334t^5 - 1540t^3 + 22t \end{cases} \tag{14}$$

Now we substitute into Equations (9):

$$\begin{cases} 10(t^2+1)^{11}x = -15t^{22} - 24t^{21} + 1065t^{20} - 36925t^{18} + 504t^{17} + 372315t^{16} + 1536t^{15} \\ \qquad -1599750t^{14} + 1680t^{13}3232810t^{12} - 3232810t^{10} - 1680t^9 + 1599750t^8 - 1536t^7 \\ \qquad -372315t^6 - 504t^5 + 36925t^4 - 1065t^2 + 24t + 15 \\ 10(t^2+1)^{11}y = 3t^{22} + 130t^{21} - 63t^{20} - 7500t^{19} - 315t^{18} + 132570t^{17} - 273t^{16} \\ \qquad -850320t^{15} + 990t^{14} + 2491300t^{13} \qquad 2730t^{12} - 3522120t^{11} \\ \qquad +2730t^{10} + 2491300t^9 + 990t^8 - 850320t^7 - 273t^6 + 132570t^5 - 315t^4 - 7500t^3 - 63t^2 \\ \qquad 130t + 3 \end{cases} \tag{15}$$

[7] The equations are presented as they are for technical reasons.

Denote by $P_1(x, y, t)$ (resp. $P_2(x, y, t)$) the polynomial obtained by subtracting right hand side of 1st (resp. 2nd) equation from its left hand side. These polynomials generate an idela $I =< P_1, P_2 >$ in $\mathbb{R}[x, y, t]$. By elimination we obtain (quickly) one polynomial in x, y of degree 22. It is too long for being written here, we obtained it within a Maple session using the *PolynomialIdeals* package.

Here is the Maple code [S6] for the session. Note that the coefficients 0.5 and 0.3 have been entered as 1/2 and 3/10 respectively, as a priori the algorithms in *PolynomialIdeals* do not like floating point.

```
restart; with(PolynomialIdeals);
trig4 := (cos(u) + sin(u)*I)^4;
cos4 := evalc(Re(trig4));
sin4 := evalc(Im(trig4));
# rational expressions for cosine and sine
cos4rat := subs(cos(u) = (-t^2 + 1)/(t^2 + 1), subs(sin(u) = 2*t/(t^2 + 1), cos4));
cos4rat := simplify(cos4rat);
sin4rat := subs(cos(u) = (-t^2 + 1)/(t^2 + 1), subs(sin(u) = 2*t/(t^2 + 1), sin4));
sin4rat := simplify(sin4rat);
trig11 := (cos(u) + sin(u)*I)^11;
cos11 := evalc(Re(trig11));
sin11 := evalc(Im(trig11));
cos11rat := simplify(subs(cos(u) = (-t^2 + 1)/(t^2 + 1),
              subs(sin(u) = 2*t/(t^2 + 1), cos11)));
sin11rat := simplify(subs(cos(u) = (-t^2 + 1)/(t^2 + 1),
              subs(sin(u) = 2*t/(t^2 + 1), sin11)));
expand(numer(sin11rat));
# the output was long - that is a way to have a shorter expression
xsat := (-t^2 + 1)/(t^2 + 1) + (1/2)*cos11rat + (3/10)*sin4rat;
xsat := simplify(xsat);
ysat := 2*t/(t^2 + 1) + (1/2)*sin11rat + (3/10)*cos4rat;
ysat := expand(ysat);
ysat := simplify(ysat);
# definition of polynomials - pay attention to capital and lowcase letters
p1 := 10*(t^2 + 1)^11*xsat;
P1 := 10*(t^2 + 1)^11*x - p1;
p2 := 10*(t^2 + 1)^11*ysat;
P2 := 10*(t^2 + 1)^11*y - p2;
K := <P1, P2>; # definition of an ideal
KE := EliminationIdeal(K, {x, y});
Gen := Generators(KE);
Gen[1];
Gen[2];
evala(Factors(Gen[1]));
```

All the commands give an almost immediate answer, elimination requests a few seconds. The command Gen[2] returns an error message; it verifies that the ideal KE is generated by one polynomial. This had to be checked, as the output is dispatched on about 30 rows. The last command checks that this polynomial is irreducible.

5.2 A non planar situation

Here too, we illustrate the process with an example. We consider the first case shown in Figure 13. We have:

$$(x, y, z) = \left(2\cos t + \frac{2}{5}\cos 3t, 2\sin t + \frac{2}{5}\sin 3t, \frac{2}{5}\cos 2t\right). \tag{16}$$

With the same method as above, based on Equations (10), and obtain the following rational parametrization for the curve:

$$\begin{cases} x = \frac{4(-3t^6+5t^4-5t^2+3)}{5(t^2+1)^3} \\ y = \frac{32t(t^4+1)}{5(t^2+1)^3} \\ z = \frac{2(t^4-6t^2+1)}{5(t^2+1)^2} \end{cases} \tag{17}$$

We transform equations (17) into polynomials

$$\begin{aligned} P_1(x,y,z) &= 5(t^2+1)^3 x + 12t^6 - 20t^4 + 20t^2 - 12 \\ P_2(x,y,z) &= 5(t^2+1)^3 y - 32t(t^4+1) \\ P_3(x,y,z) &= 5z(t^2+1)^2 - 2t^4 + 12t^2 - 2 \end{aligned}$$

which generate an ideal $J =< P_1, P_2, P_3 >$. After elimination of the parameter t, we obtain an elimination ideal JE generated by 2 polynomials $F_1(x, y, z)$ and $F_2(x, y, z)$, such that:

$$\begin{aligned} F_1(x,y,z) &= 25x^2 + 25y^2 - 100z - 104 \\ F_2(x,y,z) &= 125z^3 + 25y^2 + 250z^2 + 60z - 72 \end{aligned}$$

It was expected to find 2 polynomials, as a space curve is defined as the intersection of two surfaces, each one determined by an equation. The 2 surfaces are displayed in Figure 15. Note that

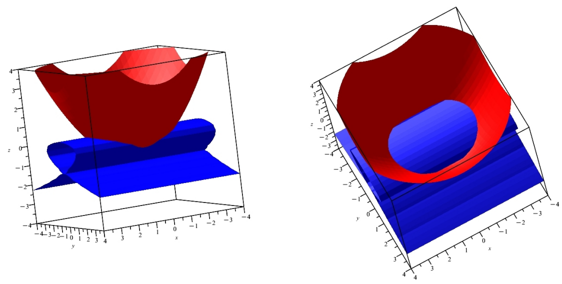

Figure 15: The trajectory for $(a, b) = (3, 2)$ with implicit equations

the intersection curve is difficult to identify. To have a good idea, the plot has to be rotated on the screen using the mouse, but even so the visual impression is not perfect. Part of the problem is the choice of the mesh to visualize surfaces in 3D space (see [31]). The plot can be enhanced if adding the parametric plot of the space curve. Figure 16 shows the respective plots obtained with the rational parametrization (on the left) and the trigonometric parametrization (on the right). The plots have been obtained using the following commands:

(a) Rational parametrization (using the same notations as above):

```
cratspace := spacecurve([xsatspace, ysatspace, zsatspace], t = -15 .. 15,
             numpoints = 5000, thickness = 3, color = black);
```

(b) Trigonometric parametrization (one period is enough):

```
ctrigplot := spacecurve([2*cos(t) + 2/5*cos(3*t),
                   2*sin(t) + 2/5*sin(3*t), 2/5*cos(2*t)],
             t = 0 .. 2*Pi, color = black, thickness = 7)
```

Note that the trigonometric parametrization gives a complete plot, but something is missing when using a rational parametrization. Filling the gap is not always possible because of issues related to limits at infinity of the involved rational functions.

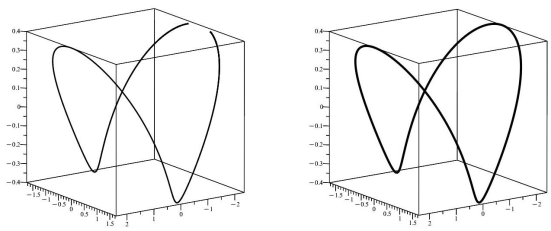

Figure 16: Plots with two parameterizations of the same curve

Figure 17 shows the surface and enhances the intersection curve.

6 Conclusions

6.1 Networking between technologies

The exploration and determination of geometric loci is an important topic in geometry. A few examples are provided by [1, 3, 2]. Our study relies strongly on the usage of two kind of software: a Dynamic Geometry Software (DGS), here GeoGebra, and a Computer Algebra System (CAS), here Maple 2024. The DGS enabled to explore the curves dynamically. It is natural to represent the dynamics of mobile points using a slider, and the trajectories are displayed dynamically (using **Trace On**, without this the points move but the user's eyes do not keep a lasting impression). Moreover, the automated command **Locus** provides complete pictures of the trajectories. Joint usage of both helps to have a graphical understanding of the phenomenon.

Nevertheless, for some parts of the study, the abilities of the DGS were not enough. Actually, it is well known that for algebraic symbolic computations, a stronger software is needed. Therefore we used Maple. In this study, we did it at two places:

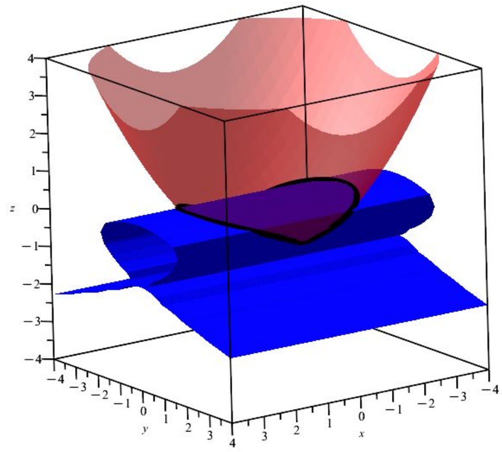

Figure 17: Visualization of the curve as intersection of two surfaces

- Implicitization involves a large amount of computations (generally transparent to the user, until the output is displayed). We have a detailed example in subsection 5.1. This is not a new phenomenon, for example it appears already in [6, 10, 11]. The computation of the elimination ideal computes actually a projection. For topological reasons (work is done according to Zariski topology), it is important to check whether the obtained polynomial is irreducible or not. It happens that irrelevant components appear, but this is not the case.

- In subsection 3.2, the exploration process is similar the dichotomy method taught to undergraduates for solving equations numerically. The DGS is mostly based on numerical methods, whence the need to add other algorithms which can check the results symbolically. This is done in the GeoGebra-Discovery companion, but not for the study here, even if we tried a more geometric construction of the mobile point Sat. Here we have an originality: generally switching from DGS to CAS is motivated by the need for symbolic computations, but here one of the switches was motivated by the need to make a numerical exploration, which could not be done totally with the other software..

In any case, we have here a new example of the importance of networking between different kinds of software. Even when both kinds provide similar features, they may have different affordances. Duval [16, 17]) explained that mathematical objects are abstract and can be approached and understood only using different registers of representation. Traditionally, the identified registers are numerical, symbolic, graphical, etc. The activities described here show that even within the same register, but with different tools, the object is presented in a different way.

An important feature of a DGS is interactivity: the possibility to drag points and the availability of sliders are central. They allow animations, with possible modifications in real time. Animations are also available with a CAS, but they require a priori programming, and the possibility to interact with the animation in real time is limited.

For a long time, wishes to develop automated networking between CAS and DGS have been expressed clearly [27, 6, 12]. In the past, it existed for certain questions between Maple and MatLab, but not on a very large scale. It is possible to export the outputs in a different language: Maple allows to export the session to rtf or LaTeX format; this is useful, but not for the sake of computations and explorations. It allows export to STL format; this is important for exploration, as this is the language used by certain 3d printers [13]. This has also limitations, and for certain problems, we had to switch to another language, without automated export [15]. Advances have been recently announced with GeoGebra-Discovery [24].

Generally, a DGS and a CAS provide different registers of representation (in the sense of Duval [16, 17]): numerical (this was almost all what was available a few decades ago), graphical, symbolic, etc. A central issue was switching between registers, most CAS enabling to switch from symbolic to graphical (generally via the numerical, but this may be transparent), sometimes in reversed direction, from graphical to numerical (in DGS), more rarely from graphical to symbolic.

6.2 Modeling and Creativity

The 4 C's of 21st Century Education are already well-known and documented, as defined in OECD documents [26, 28]: Collaboration, Communication, Critical Thinking and Creativity. The present paper is anew contribution to we emphasize two of theses C's: Communication and Creativity. As discussed in the previous subsection, Communication is understood here between machines, the man-and-machine communication being minor, and between humans not discussed at all. As a follow-up of [7] (where we explained being on the verge of mathematical art), we have here a new contribution to explore Creativity, on the basis of simple models of trajectories in space.

We enhanced the exploratory aspect of the work, which has been enabled by the sliders of the applets. It is clear that exploration request, first of all, the 5th C, namely Curiosity[8].

Actually, the initial motivation was to describe/model trajectories in space (Section 1.1). Because of the complexity of space reality, we had to switch to simple models. Generally, models are intended to provide tools for understanding the concrete situation [20]. Here we went to another direction, as "true modeling" is out of reach of the visualization abilities of our computers. Choosing arbitrary parameters may provide some understanding of concrete situations, but it leads to create exotic curves beyond what is usually presented in courses. The regular usage of modelling appears in the left part of Figure 18 (Source: [8]), we went more towards the right part.

Finally, we wish to mention that Duval [16, 17] explained that in opposition to other scientific domains, where the objects of study are present on the table or graspable, mathematical objects are abstract and can be studied only via representations, that he classifies into registers. Switching between registers enable to master the object. We did it between numerical, symbolic and graphical registers. In [13], a new register is proposed, namely 3D printed objects, which are really graspable. Their pros and cons are discussed there, such as the necessity to thicken the curve to have them 3d printed. This register is beyond the scope of the present paper.

Interactive exploration is intended to educate students to be active and more involved in their learning process than with the traditional segment definition-theorem-proof. The teacher is a mentor

[8] As we mentioned initially that we have been inspired by news from the space exploration, we cannot omit to recall that one of the NASA robots on Mars is called Curiosity.

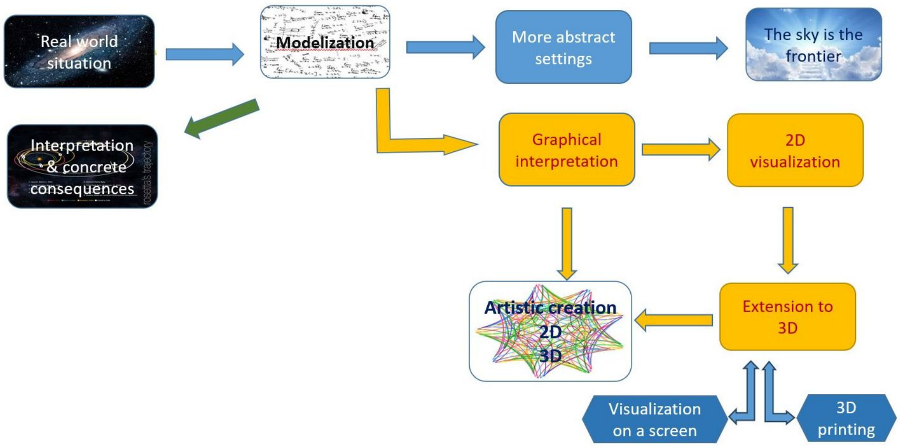

Figure 18: Modelling and creativity

and is here to facilitate the exploration, to help to understand, not to convey knowledge ex-cathedra. Nowadays, starting from real examples, taken from everyday life, from the students' culture, or from the news, not only motivates them to learn here and now, but also may improve their preparation for professional life and long life learning.

7 Supplementary Electronic Material

[S1] A GeoGebra applet for tricircular motion, in Section 2.

[S2] A GeoGebra applet for tricircular motion, one object moving in retrograde direction, in section 3.

[S3] A GeoGebra applet for a lunar polar orbit, in subsection 4.1.

[S4] A GeoGebra applet for a lunar polar orbit, in section 5.2

[S5] A GeoGebra applet to check 5-fold symmetry of a trajectory, in section 3.3.

[S6] A Maple worksheet for implicitization, in subsection 5.1.

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